The vibrational spectrum of Eden clusters in many dimensions

1985 J. Phys. A: Math. Gen. 18 L713

(http://iopscience.iop.org/0305-4470/18/12/005)

J. Phys. A: Math. Gen. 18 (1985) L713–L718. Printed in Great Britain

LETTER TO THE EDITOR

The vibrational spectrum of Eden clusters in many dimensions

Deepak Dhar†

Laboratoire de Physique Théorique et Hautes Energies‡, Université Pierre et Marie Curie, 4 Place Jussieu, 75230 Paris Cedex 05, France

Received 7 June 1985

Abstract. The configuration-averaged vibrational spectrum of Eden clusters with N sites on a d-dimensional hypercubical lattice is determined exactly in the limit $d \gg \log N \gg 1$. It is shown that the spectral dimension of these clusters is 2.

The Eden model (Eden 1961) is the simplest of growth models, and has been the subject of much investigation in recent years (Peters *et al* 1979, Rácz and Plischke 1985, Jullien and Botet 1985, Family and Vicsek 1985). It can be proved that Eden clusters are compact in any space dimension (Richardson 1973, Dhar 1985) but exact expressions for any of the cluster averages are not know even in two dimensions.

Parisi and Zhang (PZ) (1984) considered Eden clusters with N sites on a d-dimensional hypercubical lattice, and showed that in the limit $d \gg N$ the problem simplifies, and can be solved exactly. The central assumption of PZ is that the coordination number of any site is much less than the maximum coordination number ($2d$) allowed on the lattice. Since the maximum coordination number in a typical cluster of size N in the large-d limit can be shown to vary as $\log N$, the regime of validity of the PZ limit may be extended to include all values of N such that $d \gg \log N$. In this limit, each site of the cluster is a surface site (has unoccupied neighbours). Thus the behaviour of Eden clusters in this limit is expected to be quite different from the limit of infinite N at fixed d. In the latter limit, the clusters are compact. The average cluster diameter varies as $N^{1/d}$ and the fractional number of surface sites is zero. The requirement that the average cluster diameter $N^{1/d}$ be much greater than 1 is the same as $\log N \gg d$. We thus expect a crossover from PZ like small-N behaviour to the asymptotic large-N behaviour at $\log N \simeq d$. Because of its analytical solubility, the PZ limit deserves further study. It may also help in a better understanding of finite sized Eden clusters.

In the large-d limit, the probability that a cluster will contain loops goes to zero, and the clusters are tree-like. They are thus the same as Eden trees, which are defined (Dhar and Ramaswamy (DR) 1985) in any dimension d by the Eden rule (equal probability of growth at any of the perimeter sites) with the added condition that occupation of sites leading to formation of loops in the cluster is forbidden. Numerical simulations of DR showed that diffusion on these clusters is anomalous, with the spectral dimension $\tilde{d}$ (characterising the density of states of low-frequency vibrational modes of the system) not equal to d, and also not equal to $2d/d_w$ where d_w is fractal dimension of Brownian random walks on the cluster. The relation $\tilde{d} = 2d/d_w$ is

† On leave from Tata Institute of Fundamental Research, Bombay, India.
‡ Laboratoire associé au CNRS UA 280.

sometimes used to define the spectral dimension $\tilde{d}$ (e.g. Alexander and Orbach 1982) and its breakdown implies that the two definitions are not equivalent.

For Eden trees, the spectral dimension $\tilde{d}$ seems to depend on d, the dimension of the underlying lattice. For $d=2$ and 3, Monte Carlo simulations provide an estimate $\tilde{d}=1.2$ and 1.3 respectively. It has not been computed analytically for any of the growth models so far. In this letter, the vibrational spectrum of Eden clusters is determined exactly in the PZ limit $d \gg \log N \gg 1$. Besides its relationship to the problem of Eden trees in finite dimensions, the problem is of interest as it provides an *exact determination of the configuration-averaged vibrational spectrum of a system with randomness in more than one dimension.*

We then obtain the density profile of Eden clusters in the PZ limit as a function of the distance from the origin, and also the fractional number of sites having a specified number of descendants. The latter result, in conjunction with the node-counting theorem for trees, is used to derive a series representation for the configuration-averaged vibrational spectrum. We determine the low-frequency limit of the integrated spectrum and show that it corresponds to $\tilde{d}=2$.

Following PZ, we specify an $(n+1)$ site cluster in infinite dimensions by its topological graph having $(n+1)$ vertices (labelled 0 to n) with lines joining nearest neighbours. Each graph is a tree. The site labelled 0 is called the root. For each site except the root, there is exactly one neighbouring site, called its predecessor, with a smaller label. Sites which will be disconnected from the root if any particular site is removed are called its descendants. The subtree formed by a site and its descendants is called a descendant subtree.

For the rth site, there are r possible predecessors. Hence there are $N!$ distinct allowed labelled trees with $(N+1)$ sites. In the Eden model, each of these graphs is assigned an equal probability of occurrence. It is easy to determine some average structural properties of these trees.

Let $n(r, N)$ be the expected number of sites at a distance r from the root in a cluster of $(N+1)$ sites, where the distance is measured along bonds. Clearly, $n(0, N)$ is 1, and for all $N>1$

$$n(r, N)=n(r, N-1)+n(r-1, N-1)/N. \tag{1}$$

For large N, these equations may be approximated as

$$N\,\partial n(r, N)/\partial N=n(r-1, N) \tag{2}$$

which are easily integrated to give

$$n(r, N)\simeq(\log N)^r/r!. \tag{3}$$

The first moment of the number distribution function $n(r, N)$, $\langle r\rangle$, is proportional to the average extent of clusters measured along bonds. From (3) it is easy to see that $\langle r\rangle$ is equal to $\log N$ asymptotically for large N in agreement with the result of PZ. Higher moments of the distribution are also easy to compute.

One can also calculate the fractional number of sites as a function of its depth from the boundary. For example, the probability that none of the subsequently added sites gets attached to the ith site is the product of probabilities that $(i+1)$th, $(i+2)$th . . . , Nth sites do not attach to it. Hence it is equal to

$$\frac{i}{i+1}\,\frac{i+1}{i+2}\cdots\frac{N-1}{N}=\frac{i}{N}. \tag{4}$$

Hence the mean number of sites with no descendants in a tree of $(N+1)$ sites is

$$\sum_{i=1}^{N} \frac{i}{N} = \frac{(N+1)}{2}. \tag{5}$$

The probability that site i has precisely one descendant site j $(i+1 \leq j \leq N)$ is similarly seen to be

$$\left[\prod_{r=i+1}^{j-1}\left(\frac{r-1}{r}\right)\right]\frac{1}{j}\left[\prod_{s=j+1}^{N}\left(\frac{s-2}{s}\right)\right] = \frac{i}{N(N-1)}, \tag{6}$$

and is independent of j. Hence the probability that site i has exactly one descendant (any j) is $i(N-i)/N(N-1)$. Summation over i of this probability gives us the expected number of sites having precisely one descendant in the cluster. By simple algebra, it is easy to see that the fractional number of such sites is $\frac{1}{6}$ for large N.

Similarly, we can calculate the fractional number of sites whose descendant subtree is topologically equivalent to a given graph (figure 1). The probability that a site i has exactly r descendants $j_1, j_2, \ldots, j_r$ $(i<j_1<j_2\ldots<j_r \leq N)$ with a specified predecessor for each j_r is $i(N-r-1)!/N!$ independent of $j_1, j_2, \ldots, j_r$. Summing over all the allowed values of j_r's, and averaging over i, we find the average fractional number of sites whose descendants form a subtree topologically equivalent to a given graph T_r with $(r+1)$ vertices. This is the same for all graphs T_r with the same number of vertices, and is equal to $1/(r+2)!$. The number of distinct subtrees T_r with r descendants is $r!$. Hence the probability that a randomly picked site in a large tree has exactly r descendants is $1/(r+1)(r+2)$.

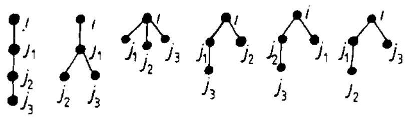


Figure 1. The six possible descendant subtrees for a site i with three descendants j_1, j_2, j_3 $(i<j_1<j_2<j_3 \leq N)$. Note that subtrees with different descent histories corresponding to the same final geometrical configuration (e.g. last three shown here) are counted as distinct.

The vibrational problem associated with the Eden clusters is defined as follows: to each occupied site of the Eden cluster, we assign a unit mass and connect all nearest-neighbour masses on the cluster by springs of unit spring-constant. The

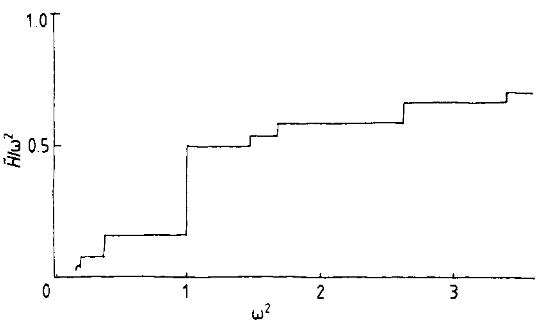


Figure 2. The integrated frequency spectrum for Eden clusters $\tilde{H}(\omega^2)$ as a function of ω^2 (schematic).

Hamiltonian of this system of coupled oscillators is written as

$$H=\sum_{\rm nn}\tfrac{1}{2}(\chi_i-\chi_j)^2+\sum_{i=1}^{N}\tfrac{1}{2}\lambda\chi_i^2 \tag{7}$$

where χ_i is the scalar displacement associated with the site i, and the first summation extends over all nearest neighbours. We have introduced a coupling constant λ for later convenience. The mass at the root is assumed held fixed so that $\chi_0\equiv 0$. The partition function of this system is given by

$$\begin{aligned}Z(\lambda)&=\prod_{i=1}^{N}\left(\int_{-\infty}^{+\infty}\frac{\mathrm{d}\chi_i}{(2\pi)^{1/2}}\right)\exp[-H]\\&=\prod_{i=1}^{N}(\omega_i^2+\lambda)^{-1/2}\end{aligned} \tag{8}$$

where ω_i ($i=1$ to N) are the frequencies of the N eigenmodes of the system when $\lambda=0$.

The frequency spectrum of this system of coupled oscillators is determined most easily by an application of the node-counting theorem for trees (DR). For each site i, other than the root, we define a function $F_i(\omega^2)$ recursively as follows.

(i) If site i has no descendants, then

$$F_i(\omega^2)=(1-\omega^2+\lambda)^{-1}. \tag{9}$$

(ii) Otherwise,

$$F_i(\omega^2)=\left(1+\lambda-\omega^2+\sum_k[1-F_k(\omega^2)]\right)^{-1} \tag{10}$$

where the summation over k extends over all the direct descendants k of i. Let $H(\omega^2)$ denote the number of eigenfrequencies ω_i^2 less than ω^2. By the node-counting theorem for trees (the arguments of DR use the free boundary conditions; for the fixed boundary condition at the root, the modification is trivial) it follows that for all ω^2

$$H(\omega^2)=\sum_{i=1}^{N}\theta(-F_i(\omega^2)) \tag{11}$$

where $\theta(\cdot)$ is the Heaviside step function.

Let $\mathscr{H}_i$ be the sub-Hamiltonian which describes the interaction between the site i and its descendants. Explicitly,

$$\mathscr{H}_i=\sum_{\rm nn}{}'\tfrac{1}{2}(\chi_j-\chi_k)^2+\tfrac{1}{2}\sum_k{}'\lambda\chi_k^2 \tag{12}$$

where the primed summations extend only over sites belonging to the descendant subtree of i. By using the recursion relation (10), and induction over the size of the descendant subtree of i, it can be shown that (details omitted) for any site i

$$\prod_j{}'\left(\int_{-\infty}^{+\infty}\frac{\mathrm{d}\chi_j}{(2\pi)^{1/2}}\right)\exp(-\mathscr{H}_i)=\prod_j{}'[F_j(\omega^2=0)]^{1/2}\exp[-\tfrac{1}{2}\{1/F_i(\omega^2=0)-1\}\chi_i^2]. \tag{13}$$

As a simple consequence of this result, putting $i=0$, and $\chi_i=0$ we get

$$Z(\lambda)=\prod_{j=1}^{N}[F_j(\omega^2=0)]^{1/2}. \tag{14}$$

$F_i(\omega^2)$ depends only on the topology of the subtree formed by the descendants of i. For all sites i with no descendants (approximately $\frac{1}{2}N$ in number for large N) we have $F_i(\omega^2) = 1/(1+\lambda-\omega^2) \equiv f_0(\omega^2)$ (say). Of the remaining sites, approximately $\frac{1}{6}N$ sites have exactly one descendant and for them

$$F_i(\omega^2) = (2+\lambda-\omega^2-f_0(\omega^2))^{-1} \equiv f_1(\omega^2) \text{ (say).}$$

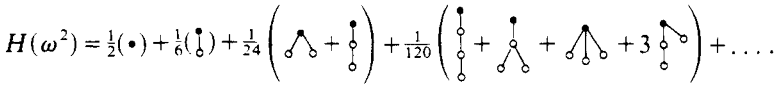
Proceeding in this fashion, grouping together sites with the same subtree structure in (11) we can write the following (exact) series representation for the normalised integrated configuration-averaged frequency spectrum

$$\tilde{H}(\omega^2) = \lim_{N\to\infty} \frac{1}{N} H(\omega^2) = \sum_{r=0}^{\infty} \frac{1}{(r+2)!} \left(\sum_{T_r} \theta(-f_{T_r}(\omega^2)) \right) \qquad (15)$$

where $f_{T_r}(\omega^2)$ is the value of $F(\omega^2)$ corresponding to a site whose descendant subtree is isomorphic to a tree graph T_r with r descendants, and the summation over T_r is the $r!$ distinct labelled subtrees with r descendants. Writing down the first few terms of this series explicitly

$$H(\omega^2) = \tfrac{1}{2}\theta(-f_0(\omega^2)) + \tfrac{1}{6}\theta(-f_1(\omega^2)) + \ldots . \qquad (16)$$

This equation has the following self-evident diagrammatic representation


(17)

Equation (16) is a series of positive terms. Though each individual term in it is not a non-decreasing function of ω^2, the summation $H(\omega^2)$ is, as is obvious from its definition. It is easily shown that as ω^2 tends to infinity, each of the $f_{T_r}(\omega^2)$ is negative. Hence the contribution of the rth-order terms to the sum is $1/\{(r+1)(r+2)\}$ for large ω^2. As

$$\sum_{r=0}^{\infty} \frac{1}{(r+1)(r+2)} = 1. \qquad (18)$$

We verify that $H(\omega^2)$ tends to 1 as ω^2 tends to infinity. The convergence of the series, though uniform, is rather slow. There are $r!$ distinct diagrams of order r (grouping together topologically equivalent subtrees with different histories, this number may be reduced to still very substantial 2^{r-1}), and the error incurred of the series is truncated after r terms decreases only as $[1/(r+2)]$.

For the remainder of this letter, we shall need the values of $F_i(\omega^2)$ only at $\omega^2 = 0$, and we shall write $F_i(\lambda)$ instead of the more cumbersome $F_i(\omega^2 = 0, \lambda)$. From (14) we get

$$\log Z(\lambda) = \tfrac{1}{2} \sum_{i=1}^{N} \log F_i(\lambda) \qquad (19)$$

and grouping together sites with topologically equivalent subtrees, as in the case of $H(\omega^2)$ earlier, we get for large N after configuration averaging

$$\lim_{N\to\infty} \log Z(\lambda)/N = \tfrac{1}{2} \sum_{r=0}^{\infty} \frac{1}{(r+2)!} \left(\sum_{T_r} \log(f_{T_r}(\lambda)) \right). \qquad (20)$$

The spectral dimension $\tilde{d}$ is defined by the relation

$$\tilde{d} = \lim_{\omega\to 0^+} \log H(\omega^2)/\log \omega. \qquad (21)$$

Then from (8) it follows that for small $\lambda > 0$ and $\tilde{d} < 2$

$$\log \tilde{Z}(\lambda) \overset{\text{def}}{=} \lim_{N\to\infty} \log Z(\lambda)/N \approx \log Z(0) + C\lambda^{d/2} + \text{higher order terms in } \lambda \tag{22}$$

and for $\tilde{d} = 2$, we get for small positive λ

$$\log Z(\lambda) = \log Z(0) + C\lambda \log \lambda + \text{higher order in } \lambda. \tag{23}$$

Differentiating (20) with respect to λ we get

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \log Z(\lambda) = \tfrac{1}{2} \sum_{r=0}^{\infty} \frac{1}{(r+2)!} \sum_{T_r} \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} \log f_{T_r}(\lambda) \right). \tag{24}$$

From the recursion equations (9) and (10) it follows that

$$f_{T_r}(\lambda = 0) = 1, \tag{25}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f_{T_r}(\lambda)|_{\lambda=0} = -(r+1). \tag{26}$$

Substituting in (24) we get

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \log \tilde{Z}(\lambda)|_{\lambda=0} = \tfrac{1}{2} \sum_{r=0}^{\infty} \frac{1}{(r+2)}, \tag{27}$$

a series which is logarithmically divergent. If λ is small but slightly greater than zero,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \log f_{T_r}(\lambda) \approx -(r+1)$$

for all $r \leqslant 1/\lambda$, and is small otherwise. In that case, the summation in (27) has an upper cutoff at $r \approx 1/\lambda$ and hence for small λ we get

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \tilde{Z}(\lambda) \approx \tfrac{1}{2} \log \lambda + \text{higher order in } \lambda. \tag{28}$$

Comparing with (23) we conclude that for Eden cluster in infinite dimensions

$$\tilde{d} = 2 \tag{29}$$

which is the promised result.

I thank Professor B Diu and the members of LPTHE for hospitality at the Université Pierre et Marie Curie.

References

Alexander S and Orbach R 1982 *J. Physique Lett.* **42** 625

Dhar D 1985 *Phys. Rev. Lett.* **54** 2058

Dhar D and Ramaswamy R 1985 *Phys. Rev. Lett.* **54** 1346

Eden M 1961 *Proc. 4th Berkeley Symp. in Mathematics, Statistics and Probability* ed J Neyman (Berkeley: University of California Press) p 233

Family F and Vicsek T 1985 *J. Phys. A: Math. Gen.* **18** L75

Jullien R and Botet R 1985 *Preprint*

Parisi G and Zhang Y C 1984 *Phys. Rev. Lett.* **53** 1791

Peters H P, Stauffer D, Hölters H P and Loewenich K 1979 *Z. Phys.* B **34** 399

Rácz Z and Plischke M 1985 *Phys. Rev.* A **31** 985

Richardson D 1973 *Proc. Camb. Phil. Soc.* **74** 515